DSP

29/10/2015 الخميس د. عرفة المحاضرة 5

Discrete Fourier Transform (D.F.T)

If we want to analyze any signal, we must evaluate some parameters as:-

1- freq. content.
2- Power and Energy Density.
3- Amplitude.
4- Periodicity.

→ Both are analyze in freq. domain

we want to convert signals from time domain to freq. domain, to study some parameters of the signals, as (freq. content and Power and Energy density), which can not be studied in time domain.

* Some applications depend on the analysis of the signals on freq. domain as filter design.

from Laplace Transform

$$x(t) \xrightarrow{L.T} X(s)$$

time domain → s-domain

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

- Cont. fourier transform (FT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

⇓ discretization

$X(K)$ Discrete fourier transform (DFT)

This doesn't work for digital Systems we need Discretization

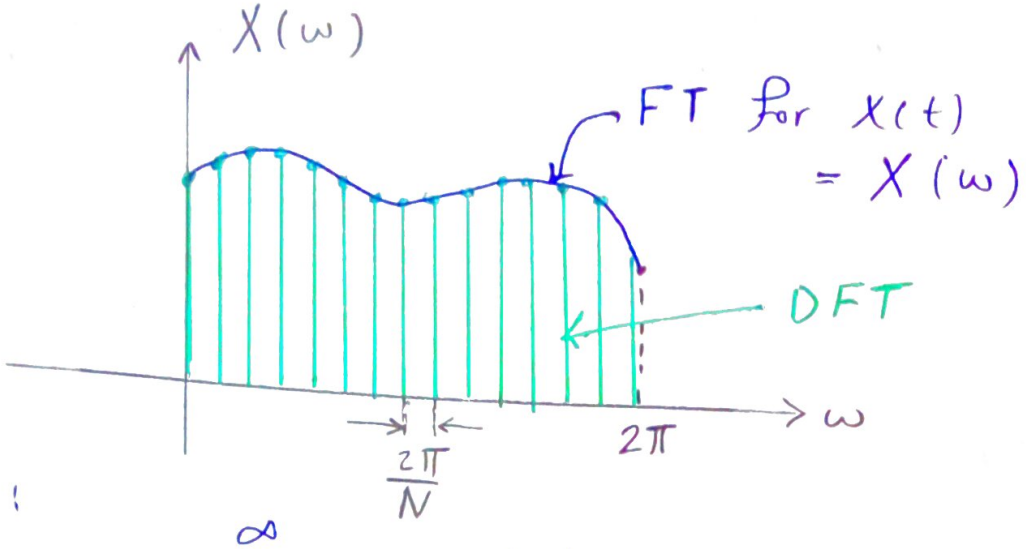
$$x(t) \xrightarrow{FT} X(\omega)$$

$$x(n) \xrightarrow{DFT} X(K)$$

* if $x(t)$ is periodic signal, then the FT of $x(t)$ $(X(\omega))$ is also Periodic. And it's periodic every 2π interval of freq. ⇐ F.T ويكون دوري

* if we assume the signals we deal with are Periodic.

⟹ Turn over



FT:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

assume $X(\omega)$ is dicretized to N samples

$$\omega \longrightarrow \frac{2\pi}{N} K$$

K: the sample number for DFT.

$K \Rightarrow [0 \rightarrow N-1]$

N: No. of samples for DFT.

DFT:-

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N} K n}$$

n: The sample no. for discrete time sequence

$x(n)$, $n = 0, 1, 2, \ldots N-1$

⟹ Turn Over

Ex: $x(n) = \{0, 1, 2, 3\}$, find the 4-point DFT of $x(n)$, given $N = 4$ $\quad n = 0, 1, 2, \ldots, N-1$ $\quad k = 0, 1, 2, \ldots, N-1$

$$k = 0 \Rightarrow X(0) = \sum_{n=0}^{3} x(n)$$

$$= x(0) + x(1) + x(2) + x(3) = 6$$

$$k = 1 \Rightarrow X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{4} k n}$$

$$= x(0) + x(1)\, e^{-j\pi/2} + x(2)\, e^{-j\pi} + x(3)\, e^{-j3\pi/2}$$

$$= 0 + e^{-j3\pi/2} + 2e^{-j\pi} + 3e^{-j3\pi/2}$$

$$= \cos(90) - j\sin(90) + 2(\cos(2\pi) - j\sin(2\pi)) + 3\left(\cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right)\right)$$

$$= -2 + j2$$

$$k = 2 \Rightarrow X(2) = \sum_{n=0}^{3} x(n)\, e^{j\pi n}$$

$$= x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi}$$

$$= \cos\pi - j\sin\pi + 2(\cos(2\pi) - j\sin(2\pi)) + 3(\cos(3\pi) - j\sin(3\pi))$$

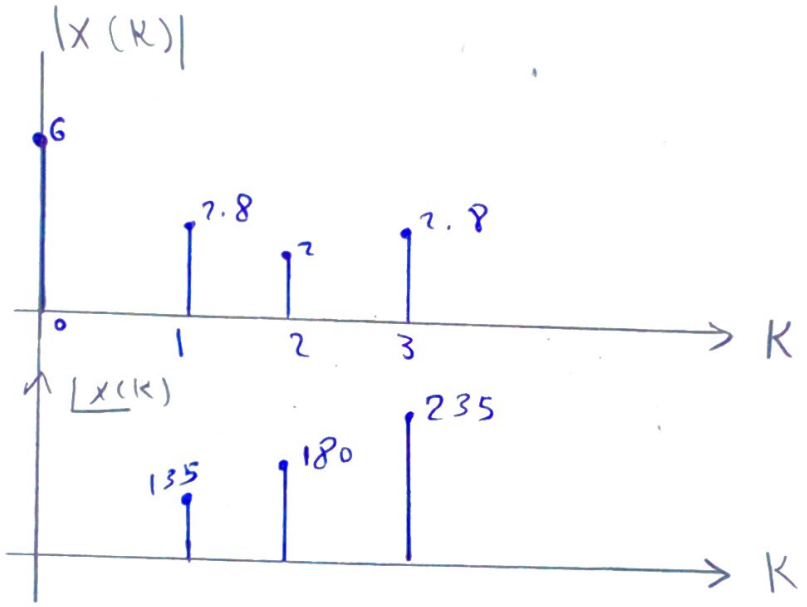
$$= -1 + 2 - 3 = -2$$

$$k = 3 \Rightarrow X(3) = -2 - 2j$$

$$X(K) = \{ 6,\ -2+j2,\ -2,\ -2-j2 \}$$

$$|X(K)| = \{ 6,\ 2\sqrt{2},\ 2,\ 2\sqrt{2} \}$$

$$\angle X(K) = \{ 0,\ 135^\circ,\ 180^\circ,\ 225^\circ \}$$



Max k for exam 3 or 4

طريقة أسرع ومفيدة Matrix

$$x(n) \xrightarrow{DFT} X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\Rightarrow X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

$K = 0, 1, 2, \dots, N-1$

$n = 0, 1, 2, \dots, N-1$

$$X_N = \begin{pmatrix} X(k=0) \\ X(k=1) \\ \vdots \\ X(N-1) \end{pmatrix}, \quad x_N = \begin{pmatrix} x(n=0) \\ x(n=1) \\ \vdots \\ x(n=n-1) \end{pmatrix}$$

$$X_N = \begin{matrix} n=0 & n=1 & \cdots & n=N-1 \\ k=0 \\ k=1 \\ \vdots \\ k=N-1 \end{matrix} \begin{bmatrix} & & & \\ & W_N^{kn} & & \\ & & & \end{bmatrix} x_N$$

$N \times N$

Ex: $N = 4$

$$\left[W_4^{kn} \right] = \begin{matrix} & n=0 & n=1 & n=2 & n=3 \\ k=0 & W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ k=1 & W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ k=2 & W_{61}^0 & W_4^2 & W_4^4 & W_4^6 \\ k=3 & W_4^0 & W_4^3 & 4_4^6 & W_4^9 \end{matrix}$$

$$W_4 = e^{-j\frac{2\pi}{4}} = \cos(90) - j\sin(90) = -j$$

$$W_4^0 = 1$$

$$W_4^2 = e^{-j\pi} = \cos(180) - j\sin(180) = -1$$

$$W_4^3 = e^{-j\frac{3}{2}\pi} = \cos(270) - j\sin(270) = j$$

$$W_4^4 = W_4^0 = 1$$

$$W_4^6 = W_4^2 = -1$$

$$W_4^9 = W_4^1 = -j$$

$$W_N^N = e^{\frac{-j2\pi N}{N}} = e^{-j2\pi} = 1$$

$$W_4^6 = W_4^2 \cdot W_4^4$$

$$W_4^9 = W_4^4 \cdot W_4^4 \cdot W_4^1$$

⇒ Turn over

$$[W_4] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

DFT for $x(n)$: $(N=4)$

$$X_4 = [W_4]\, x_4$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix}$$

(6 : $X(K=0)$; $-2-2j$: $X(K=4)$)

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for $\boxed{N=3}$

$$[W_3]_{3\times 3} = \begin{array}{c} \\ K=0 \\ K=1 \\ K=2 \end{array} \begin{array}{c} \begin{array}{ccc} n=0 & n=1 & n=2 \end{array} \\ \begin{pmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{pmatrix} \end{array}$$

$W_3^0 = 1$

$W_3^1 = e^{-j\frac{2\pi}{3}} = \cos(120^\circ) - j\sin(120^\circ) = -0.5 - j\sqrt{3}/2$

$W_3^2 = e^{-j\frac{4\pi}{3}} = \cos(240^\circ) - j\sin(240^\circ) = -0.5 + j\sqrt{3}/2$

$W_3^4 = W_3^1 = -0.5 - j\sqrt{3}/2$

$$[W_3] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -0.5-j\sqrt{3}/2 & -0.5+j\sqrt{3}/2 \\ 1 & -0.5+j\sqrt{3}/2 & -0.5-j\sqrt{3}/2 \end{pmatrix}$$

Twiddle Factor

Ex: $x(n) = \{0.5, 1\}$, find the 3-point DFT

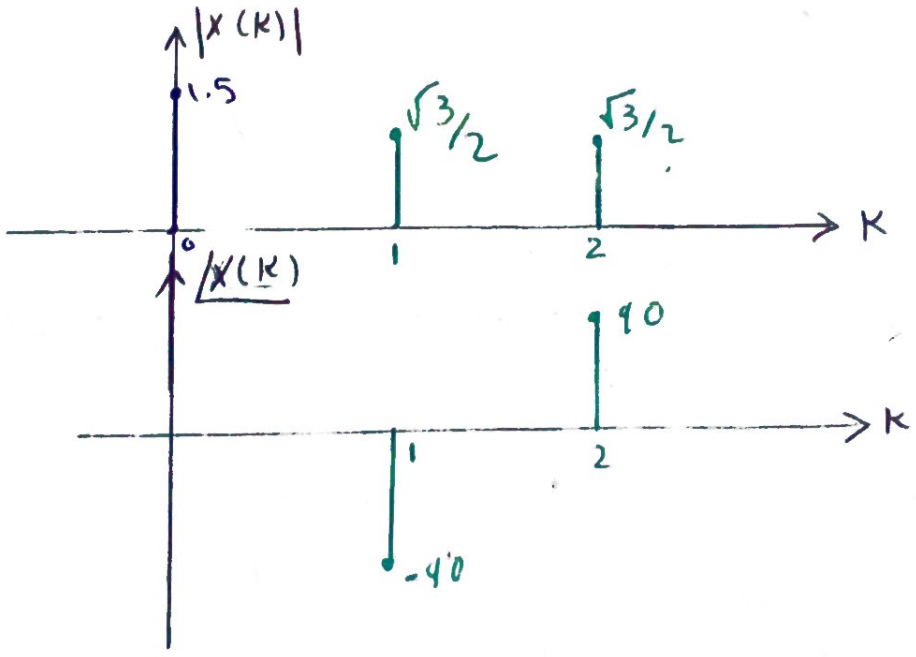
$N = 3$

DFT $X(K) \Longrightarrow X_3 = [W_3]_{3\times 3}\, x_3$

$$X_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5 - j\sqrt{3}/2 & -0.5 + j\sqrt{3}/2 \\ 1 & -0.5 + j\frac{\sqrt{3}}{2} & -0.5 - j\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 \\ -j\sqrt{3}/2 \\ j\sqrt{3}/2 \end{bmatrix}$$

$$X(K) = \{1.5, -j\sqrt{3}/2, j\sqrt{3}/2\}$$

$$|X(K)| = \{1.5, \sqrt{3}/2, \sqrt{3}/2\}$$

$$\angle X(K) = \{0, -90, 90\}$$